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STORAGE THEORY AND THE SUPREMA OF CERTAIN INFINITELY DIVISIBLE PROCESSES

by

J. MICHAEL HARRISON

TECHNICAL REPORT NO. 63

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FOR THE OFFICE OF NAVAL RESEARCH

Frederick S. Hillier, Project Director

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STORAGE THEORY AND THE SUPREMA OF CERTAIN

INFINITELY DIVISIBLE PROCESSES

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1. Introduction

Let $X = \{X(t),\ t \geq 0\}$ be a process with stationary, independent increments (an infinitely divisible process) and

$$-E[X(t)] = \mu t, \qquad \text{where } 0 < \mu < \infty ,$$

$$\mathrm{Var}[X(t)] = \sigma^2 t, \qquad \text{where } 0 < \sigma^2 < \infty ,$$

for $t \geq 0$. We assume that X is continuous in probability with paths that are right continuous and have left limits. We further assume that X has no negative jumps, so the Laplace transform of $X(t)$ exists and has the form

(1) $$E[e^{-\alpha X(t)}] = e^{\Phi(\alpha)t} \qquad \text{for} \quad \alpha > 0 \quad \text{and} \quad t \geq 0.$$

The exponent function $\Phi(\alpha)$ can be uniquely represented in the form

(2) $$\Phi(\alpha) = \mu\alpha + \sigma^2 \int_{[0,\infty)} x^{-2}(e^{-\alpha x} - 1 + \alpha x)\, G(dx) ,$$

where $G(\cdot)$ is a probability distribution on $[0,\infty)$ satisfying

(3) $$\int_{(0,\infty)} x^{-1}\, G(dx) < \infty ,$$

cf. Gnedenko (1968), pp. 323-328. From (2) and (3) it follows easily that $\Phi(\cdot)$ is convex and strictly increasing with

(4) $$\Phi(0) = 0, \qquad \Phi'(0) = \mu \qquad \text{and} \qquad \Phi''(0) = \sigma^2 .$$

Thus for each $\alpha > 0$ there exists a unique $\omega(\alpha) > 0$ such that

(5) $$\Phi(\omega(\alpha)) = \alpha \qquad (\alpha > 0) .$$

We define

$$M(t) = \sup\{X(s): 0 \le s \le t\}, \qquad m(t) = -\inf\{X(s): 0 \le s \le t\}$$

for $t \ge 0$. It is known that $-X(t)/t \to \mu$ almost surely as $t \to \infty$, so $M = \lim M(t)$ is almost surely finite. Let $H(\cdot)$ be the distribution function (d.f.) of M. We define the first passage times (recall that X has no negative jumps)

$$\theta(x) = \inf\{t \ge 0: X(t) + x = 0\} , \qquad x \ge 0$$

The following proposition was proved by Zolotarev (1964) using analytical methods. An alternate proof, using combinatorial methods, is given by Takacs (1967), p. 88. Yet another (and I think simpler) proof is easily constructed from the well known fact that

$$Z(t) = e^{-\omega(\alpha)X(t)-\alpha t} , \qquad t \ge 0 ,$$

is a martingale.

Proposition 1: $E[e^{-\alpha\theta(x)}] = e^{-\omega(\alpha)x}$ for $x \geq 0$ and $\alpha > 0$.

The central purpose of this paper is to prove by elementary means that the Laplace transform of M is given by $\mu\alpha/\Phi(\alpha)$. This beautiful result, which greatly generalizes the famous Pollaczek-Khinchine formula of queuing theory, is also due to Zolotarev (1964), and again an alternate combinatorial proof has been given by Takacs (1967), p. 86. Our approach relies heavily on the close relationship between the maximum process $\{M(t), t \geq 0\}$ and the storage process $W = \{W(t), t \geq 0\}$ defined by

$$(6) \qquad W(t) = \sup\{X(t) - X(s): 0 \leq s \leq t\} = X(t) + m(t), \qquad t \geq 0$$

Speaking loosely, we say that W is obtained from X by the imposition of a reflecting barrier at zero. The following observation has been made by Gani and Prabhu (1960) and by others. It follows immediately from the fact that, for each fixed $t > 0$, $\{X(t) - X(t-s), 0 \leq s \leq t\}$ has the same distribution as $\{X(s), 0 \leq s \leq t\}$. (Each has stationary, independent increments, and the one-dimensional distributions are the same.)

Proposition 2. For each $t \geq 0$, $W(t)$ and $M(t)$ have the same distribution.

From this we see that $W(t) \Rightarrow M$ as $t \to \infty$, meaning that $H(\cdot)$ is the limit distribution of W. In section 2 we study the Markov process W, obtaining a simple formula for the Laplace transform of $E[W(t)]$. Zolotarev's result is then easily gotten from the fact that $H(\cdot)$ is the unique stationary distribution of W. Our analysis requires the assumption that $X(t)$ has finite variance, a restriction not imposed in Zolotarev's

original treatment. In section 3 we show how the general result can be gotten from that for the case of finite variance.

2. The Storage Process

Generalizing (6), we define the storage process W with initial state $x \geq 0$ by

$$W(t) = x + X(t) + [m(t) - x]^+ , \qquad t \geq 0 . \tag{7}$$

For each $t \geq 0$, let

$$X_t(s) = X(t + s) - X(t) \qquad s \geq 0 ,$$

$$m_t(s) = - \inf\{X_t(u):0 \leq u \leq s\}, \qquad s \geq 0 .$$

With $W(\cdot)$ defined by (7), the reader may verify that

$$W(t + s) = W(t) + X_t(s) + [m_t(s) - W(t)]^+$$

for $t \geq 0$ and $s \geq 0$. Comparing this with (7), we see that W is a Markov process with stationary transition probabilities and state space $S = [0,\infty)$. In the usual way, we denote by $P_x(\cdot)$ the probability distribution on the path space of W corresponding to an initial state of x. The corresponding expectation operator is denoted by $E_x(\cdot)$. We assume that the underlying process X is defined on some probability space $(\Omega,\mathcal{F},P)$. Thus the distribution P_x is induced from P by the mapping (7).

We denote by $E(\cdot)$ the expectation operator corresponding to P.

Let $F_x(t) = P\{\theta(x) \leq t\}$ and observe that $P\{m(t) \geq x\} = F_x(t)$ for $t \geq 0$ and $x \geq 0$. Thus

$$E\{[m(t) - x]^+\} = \int_0^\infty P\{m(t) - x \geq y\}\, dy$$

$$= \int_x^\infty P\{m(t) \geq y\} dy = \int_x^\infty F_y(t)\, dy.$$

From this and Fubini's Theorem we then obtain

$$(8) \qquad \int_0^\infty e^{-\alpha t} E\{[m(t) - x]^+\}\, dt = \int_x^\infty \int_0^\infty e^{-\alpha t} F_y(t)\, dt\, dy .$$

Now, using Fubini's Theorem and Proposition 1, we have

$$(9) \qquad \int_0^\infty e^{-\alpha t} F_y(t)\, dt = \int_0^\infty c^{-\alpha t} \int_0^t F_y(du)\, dt$$

$$= \int_0^\infty \int_u^\infty e^{-\alpha t} dt\; F_y(du) = \frac{1}{\alpha} \int_0^\infty e^{-\alpha u} F_y(du)$$

$$= \frac{1}{\alpha} F[e^{-\alpha\theta(y)}] = \frac{1}{\alpha} e^{-\omega(\alpha)y} .$$

Combining (8) and (9) gives

$$(10) \qquad \int_0^\infty e^{-\alpha t} E\{[m(t) - x]^+\}\, dt = \frac{1}{\alpha} \int_x^\infty e^{-\omega(\alpha)y}\, dy = e^{-\omega(\alpha)x}/\alpha\omega(\alpha) .$$

Now for $x \geq 0$ and $\alpha > 0$ we define

$$\varphi_x(\alpha) = \int_0^\infty e^{-\alpha t} E_x[W(t)]\, dt .$$

From (7) and (10) it follows immediately that

$$\varphi_x(\alpha) = x/\alpha - \mu/\alpha^2 + e^{-\omega(\alpha)x}/\alpha\omega(\alpha) . \tag{11}$$

If $X(t) = Y(t) - ct$, where Y is an additive process and c is a positive constant, then $W(t)$ can be interpreted as the content of a dam at time t. The starting state x represents the initial content, Y represents the input process to the dam, and c represents the (constant) release rate. (See Gani and Prabhu (1960) or Takacs (1967), Chapter 6.) Thus formula (11) is of considerable interest in itself, giving us the Laplace transform of the expected content of the dam as a function of the initial content. It has other uses as well, however. From Proposition 2 we know that $E_0[W(t)] = E[M(t)]$ for all $t \geq 0$, and thus

$$f_0(\alpha) = \int_0^\infty e^{-\alpha t} E[M(t)]\, dt \qquad \text{for } \alpha > 0 .$$

Since $E[M(t)] \nearrow E(M)$ as $t \nearrow \infty$, it follows immediately that $\alpha f_0(\alpha) \nearrow E(M)$ as $\alpha \searrow 0$. Differentiating (5) twice and using (4), we obtain

$$\omega(0) = 0, \qquad \omega'(0) = 1/\mu \qquad \text{and} \qquad -\omega''(0) = \sigma^2/\mu^3 . \tag{12}$$

Setting $x = 0$ in (11), letting $\alpha \searrow 0$ and using L'Hospital's rule, we find that $\alpha f_0(\alpha) \to \sigma^2/2\mu$ as $\alpha \searrow 0$. Thus we have the following

<u>Proposition 3</u>. $E(M) = \int_S x\, H(dx) = \sigma^2/2\mu$.

Observe that $W(\theta(x)) = 0$ if $W(0) = x$. Also, X has the strong Markov property, cf. Hunt (1956). Since $\theta(x)$ is a Markov time for X, we then have from (7) and Proposition 2

$$P_x\{W(\theta(x) + t) \leq y\} = P_0\{W(t) \leq y\} = P\{M(t) \leq y\}$$

for $t \geq 0$, $x \geq 0$ and $y \geq 0$. Since $\theta(x)$ is almost surely finite and $P\{M(t) \leq y\} \to H(y)$ as $t \to \infty$, it follows easily from this that

$$(13) \qquad P_x\{W(t) \leq y\} \to H(y) \quad \text{as} \quad t \to \infty \quad \text{for all} \quad x \geq 0 \ .$$

When a Markov process W has a limit distribution $H(\cdot)$, independent of the initial state, it is well known that $H(\cdot)$ is also a stationary distribution for W, meaning in our case that

$$(14) \qquad \int_S P_x\{W(t) \leq y\}\, H(dx) = H(y) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad y \geq 0.$$

The proof of this general proposition for Markov processes is virtually identical to that of the corresponding result for Markov chains, cf. Breiman (1968), pp. 134-135. (Furthermore, $H(\cdot)$ is the unique stationary distribution, but we do not actually need this fact.) From (14) and Proposition 3 it follows that

$$\int_S E_x[W(t)]\, H(dx) = \sigma^2/2\mu \qquad \text{for all} \quad t \geq 0.$$

Using this, the definition of $\varphi_x(\alpha)$, and Fubini's Theorem, we then have

(15) $$\int_S \varphi_x(\alpha)\, H(dx) = \int_0^\infty e^{-\alpha t} \int_S E_x[W(t)]\, H(dx)\, dt$$

$$= \int_0^\infty e^{-\alpha t}\, (\sigma^2/2\mu)\, dt = \sigma^2/2\mu\alpha \ .$$

But, directly from (11) and Proposition 3,

(16) $$\int_S \varphi_x(\alpha)\, H(dx) = \sigma^2/2\mu\alpha - \mu/\alpha^2 + [\int_S e^{-\omega(\alpha)x}\, H(dx)]/\alpha\omega(\alpha) \ .$$

Combining (15) and (16), we have

$$\int_S e^{-\omega(\alpha)x}\, H(dx) = \mu\omega(\alpha)/\alpha \ .$$

Replacing α by $\Phi(\alpha)$, and hence $\omega(\alpha)$ by α, gives the central result.

<u>Proposition 4</u>. $E(e^{-\alpha M}) = \int_S e^{-\alpha x}\, H(dx) = \mu\alpha/\Phi(\alpha)$ for $\alpha > 0$.

3. <u>The Case of Infinite Variance</u>

Suppose that X is as described in section 1 except that $\mathrm{Var}[X(t)] = \infty$. The Laplace transform of $X(t)$ still exists and has the form (1), but the exponent function $\Phi(\alpha)$ cannot be represented in the form (2). For each $k = 1, 2, \ldots$ let $X_k = \{X_k(t), t \geq 0\}$ be obtained from X by truncating all jumps of size greater than k (replacing each by a jump of size k). Then $\mathrm{Var}[X_k(t)] < \infty$. We denote by μ_k, $\Phi_k(\alpha)$ and M_k the negative of the mean, the exponent function, and the supremum respectively of X_k. For each $t \geq 0$, $X_k(t) \nearrow X(t)$ a.s.

as $k \nearrow \infty$. Thus $\mu_k \searrow \mu$, $\Phi_k(\alpha) \searrow \Phi(\alpha)$ for all $\alpha > 0$, and $M_k \nearrow M$ a.s. as $k \nearrow \infty$. Combining these facts with Proposition 4 and monotone convergence, we have

$$E(e^{-\alpha M_k}) = \mu_k \alpha / \Phi_k(\alpha) \to \mu\alpha/\Phi(\alpha) = E(e^{-\alpha M})$$

as $k \to \infty$. This same truncation argument can be used to show that formula (11) also remains valid when $\mathrm{Var}[X(t)] = \infty$.

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Abstract

Let $X = \{X(t), t \geq 0\}$ be a process with stationary, independent increments and no negative jumps. Let $W = \{W(t), t \geq 0\}$ be this same process modified by a reflecting barrier at zero (a storage process). Assuming that $-E[X(t)] = \mu t > 0$, let $M = \sup\{X(t): t \geq 0\}$, and denote by $\phi(\alpha)$ the exponent function of X. A simple formula is derived for the Laplace transform of $E[W(t)]$, $t \geq 0$, as a function of $W(0)$. Using the fact that the distribution of M is the unique stationary distribution of the Markov process W, this yields an elementary proof that the Laplace transform of M is $\mu\alpha/\phi(\alpha)$. If $\mathrm{Var}[X(t)] = \sigma^2 t < \infty$, it follows that $E(M) = \sigma^2/2\mu$. These surprisingly simple formulas were originally obtained by Zolotarev using analytical methods.

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